3167. Proposed by Arkady Alt, San Jose, CA, USA.

Let ABC be a non-obtuse triangle with circumradius R. If a,b,c are the lengths of the sides

opposite angles A,B,C, respectively, prove that

$$a\cos^3 A + b\cos^3 B + c\cos^3 C \le \frac{abc}{4R^2}.$$

Solution.

Let F be area of the triangle and let $\Delta(x,y,z) = 2xy + 2yz + 2zx - x^2 - y^2 - z^2$.

Since $abc = 4FR$ and $16F^2 = \Delta(a^2,b^2,c^2)$ then $a\cos^3 A + b\cos^3 B + c\cos^3 C \le \frac{abc}{4R^2} \iff$

$\sum a\left(\frac{b^2+c^2-a^2}{2bc}\right)^3 \le \frac{abc}{4R^2} \iff \sum_{cyc} a^4(b^2+c^2-a^2)^3 \le \frac{8a^4b^4c^4}{4R^2} = \frac{2a^4b^4c^4}{R^2} =$

$\frac{32F^2R^2a^2b^2c^2}{R^2} = 32F^2a^2b^2c^2 = 2\Delta(a^2,b^2,c^2)a^2b^2c^2.$

Thus, inequality of the problem is equivalent to inequality

(**1**) $\sum_{cyc} a^4(b^2+c^2-a^2)^3 \le 2\Delta(a^2,b^2,c^2)a^2b^2c^2.$

Denoting $x := \frac{b^2+c^2-a^2}{2}$, $y := \frac{c^2+a^2-b^2}{2}$, $z := \frac{a^2+b^2-c^2}{2}$, $p := xy+yz+zx$, $q := xyz$ and assuming $a^2+b^2+c^2 = 2$ (due homogeneity of (**1**)) we obtain $a^2 = 1-x, b^2 = 1-y, c^2 = 1-z$ where $x,y,z \ge 0$ and $x+y+z = 1$.

In p,q notation we have $a^2b^2c^2 = p-q, a^2b^2+b^2c^2+c^2a^2 = 1+p$, $\Delta(a^2,b^2,c^2) = 4\left(a^2b^2+b^2c^2+c^2a^2\right) - (a^2+b^2+c^2)^2 = 4p$,

$\sum_{cyc} a^4(b^2+c^2-a^2)^3 = 8\sum_{cyc}(1-x^2)x^3 = 8\left(\sum_{cyc} x^3 - 2\sum_{cyc} x^4 + \sum_{cyc} x^5\right) =$

$8((1+3q-3p) - 2(1+4q-4p+2p^2) + (1+5q-5p+5p^2-5pq)) = 8p(p-5q)$ and then inequality (**1**) becomes $8p(p-5q) \le 8p(p-q) \iff 0 \le q$ with equality if $xyz = 0$.